

- What is the asymptotic (limiting) distribution of

$$U_n(t) = \sum_{i=1}^n \int_0^t H_i(u) dM_i(u)$$

- Answer: Martingale Central Limit Theorem (MCLT)

- $U_n(t) \Rightarrow U$  on  $(D[0, \tau], \mathcal{S})$

- $U$  is a time-transformed Brownian motion

1.  $EU(t) = 0$

2.  $\text{var}\{U(t)\} = \lim_{n \rightarrow \infty} \langle U_n, U_n \rangle (t)$

3. independent increment:  $U(s)$  and  $U(t) - U(s)$  are independent for  $s \leq t$

- Assumptions

- convergent variance:  $\langle U_n, U_n \rangle (t) \rightarrow v(t)$

- smoothness (no spikes):  $\langle U_{n,\epsilon}, U_{n,\epsilon} \rangle (\tau) \rightarrow 0$  for any  $\epsilon > 0$ , where

$$U_{n,\epsilon}(t) = \sum_{i=1}^n \int_0^t H_i(u) I(|H_i(u)| \geq \epsilon) dM_i(u)$$

- Asymptotics of Nelson-Aalen estimator

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$$U_n(t) = \sqrt{n}[\widehat{\Lambda}(t) - \Lambda(t)] = \sum_{i=1}^n \int_0^t \frac{\sqrt{n}}{Y(u)} dM_i(u)$$

for  $t < \tau = \sup_t \{t; \Pr(X \geq t) > 0\}$

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$$\begin{aligned} \langle U_n, U_n \rangle (t) &= \sum_i \int_0^t \frac{n}{Y(u)^2} Y_i(u) \lambda(u) du \\ &= \int_0^t \frac{n}{Y(u)^2} Y(u) \lambda(u) du \\ &= \int_0^t \frac{\lambda(u) du}{Y(u)/n} \longrightarrow v(t) = \int_0^t \frac{\lambda(u) dt}{EY_1(u)} \end{aligned}$$

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$$\begin{aligned} \langle U_{n,\epsilon}, U_{n,\epsilon} \rangle (t) &= \int_0^t \frac{n}{Y(u)} I\left(\frac{\sqrt{n}}{Y(u)} \geq \epsilon\right) \lambda(u) du \\ &= \int_0^t \frac{1}{Y(u)/n} I\left(\frac{n^{-1/2}}{Y(u)/n} \geq \epsilon\right) \lambda(u) du \longrightarrow 0 \end{aligned}$$

- Example: calculate  $E[dN(t) | \mathcal{F}_{t-}]$

1.  $dN(t) = 0$  or  $1$

2.  $E[dN(t) | \mathcal{F}_{t-}] = \Pr\{dN(t) = 1 | \mathcal{F}_{t-}\}$

3. if  $Y(t) = 0 \implies Y(t-) = 0$ , then  $\Pr\{dN(t) = 1 | \mathcal{F}_{t-}\} = 0$

4. if  $Y(t) = 1 \implies Y(t-) = 1$ , then

$$\Pr\{dN(t) = 1 | \mathcal{F}_{t-}\} = \Pr\{t \leq X \leq t + dt, \Delta = 1 | X \geq t\}$$

$X = \min(T, C) \geq t \Leftrightarrow T \geq t, C \geq t$ , if  $T$  and  $C$  are independent, then

$$\Pr\{dN(t) = 1 | \mathcal{F}_{t-}\} = \Pr\{t \leq T \leq t + dt | T \geq t\} = \lambda(t)dt$$

5.  $E[dN(t) | \mathcal{F}_{t-}] = Y(t)\lambda(t)dt$

6.  $M(t) = N(t) - \int_0^t Y(u)\lambda(u)du$  is a martingale

- What can we estimate?

1. At-risk probability:  $\Pr\{X \geq t\} = E[I(X \geq t)] = E[Y(t)]$

2. sub-distribution functions

	$\Delta = 0$	$\Delta = 1$
$X \leq t$	$F_0^s(t) = \Pr\{X \leq t, \Delta = 0\}$	$F_1^s(t) = \Pr\{X \leq t, \Delta = 1\}$
$X \geq t$	$S_0^s(t) = \Pr\{X \geq t, \Delta = 0\}$	$S_1^s(t) = \Pr\{X \geq t, \Delta = 1\}$
	$\lambda_0^s(t) = dF_0^s(t)/S(t)dt$	$\lambda_1^s(t) = dF_1^s(t)/S(t)dt$

- noninformative censoring

- $\lambda_1^s(t) = \lambda(t) \Rightarrow$

$$\begin{aligned} \exp \left\{ - \int_0^t S(u)^{-1} dF_1^s(t) \right\} &= \exp \left\{ - \int_0^t \lambda_1^s(u) du \right\} \\ &= \exp \left\{ - \int_0^t \lambda(u) du \right\} = S(t) \end{aligned}$$

- independent censoring

- whenever  $\Pr\{X > t\} > 0$

$$\begin{aligned}\lambda_1^s(t)dt &= \Pr\{t \leq X \leq t + dt, \Delta = 1 \mid X \geq t\} \\ &= \Pr\{t \leq T \leq t + dt, T \leq C \mid T \geq t, C \geq t\} \\ &= \Pr\{t \leq T \leq t + dt, t \leq C \mid T \geq t, C \geq t\} \\ &= \lambda_1(t)dt\end{aligned}$$

- independent censoring implies noninformative censoring